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Explicit Results for the Best Uniform Rational Approximation to Certain Continuous Functions

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A class of continuous functions is defined, and the best uniform rational approximations to these functions are then given explicitly. From these general results, we recover some particular ones on best uniform polynomial and rational approximations previously given by Bernstein, Boehm, and Rivlin.

1. INTRODUCTION

Let $f(x)$ be a continuous function on the interval $[-1, 1]$, and $V_{m,n}$ the set of all rational functions of the form

$$r(m, n; x) = p_m(x)/q_n(x), \tag{1.1}$$

where $p_m(x)$ and $q_n(x)$ are polynomials of degree $\leqslant m$ and $\leqslant n$, respectively. We define

$$E_{m,n}(f) = \inf_{V_{m,n}} \sup_{x\in[-1,1]} |f(x) - r(m, n; x)|. \tag{1.2}$$

For every pair of nonnegative integers m, n, it is well known that there exists a unique rational function $r^*(m, n; x)$ in $V_{m,n}$ such that

$$E_{m,n}(f) = \max_{x\in[-1,1]} |f(x) - r^*(m, n; x)| \tag{1.3}$$

(see Achieser [1], Chap. 2). $r^*(m, n; x)$ is called the rational function of "best uniform approximation" to $f(x)$ on $[-1, 1]$ with respect to $V_{m,n}$. Let us now denote by $\bar{r}(m, n; x)$ a rational function of the form (1.1) whose highest coefficients of $p_m(x)$ and $q_n(x)$ are nonzero. We then have the following characterization of the best uniform rational approximation.

LEMMA 1.1. *$\bar{r}(m, n; x)$ is the best uniform rational approximation to $f(x)$ with respect to $V_{m+u,n+v}$ if and only if the difference $f(x) - \bar{r}(m, n; x)$ attains its extreme values alternately in at least $(m + n + 2 + t)$ points, where $0 \leqslant u, v \leqslant t$.*

This result follows directly from a theorem due to Chebyshev (see Achieser [1], Chap. 2).

There are very few examples of functions f for which $r^*(m, n; x)$ and the maximum error $E_{m,n}(f)$ can be given explicitly (see Achieser [1], p. 66, and Boehm [2]). The purpose of this paper is to obtain $r^*(m, n; x)$ and $E_{m,n}$ for a certain class of continuous functions which we shall define in Sec. 2. The main results in Secs. 3 and 4 can be considered as extensions of some results for polynomial and rational approximations given by Rivlin [5], Bernstein (see Golomb [3]), and Boehm [2].

2. A Class of Continuous Functions

We shall begin with some definitions.

Definition 2.1. Let a be any integer and t be any real or complex number such that $|t| < 1$. For $\theta \in [0, \pi]$, we define a function $\delta(a, t; \theta)$ by

$$\delta(a, t; \theta) = i \log[(te^{ia\theta} - 1)/(t - e^{ia\theta})], \tag{2.1}$$

where we choose that branch of the logarithmic function so that $\delta(a, t; 0) = 0$ and $\delta(a, t; \pi) = a\pi$.

The function $\delta(a, t; \theta)$ has the following properties.

Lemma 2.1. (i) *$\delta(a, t; \theta)$ is a continuous function in* $[0, \pi]$, (ii) $\overline{\delta(a, t; \theta)} = \delta(a, \bar{t}; \theta)$, (iii) *$\delta$ is real for all θ in* $[0, \pi]$ *when t is real.*

Proof. Since $|t| < 1$, $\delta(a, t; \theta)$ has no singularities in $[0, \pi]$, hence it is a continuous function. (ii) and (iii) can be obtained directly from (2.1).

Definition 2.2. Let T_l be a set of l real or complex numbers $\{t_1, t_2, ..., t_l\}$ satisfying the conditions

(i) $|t_s| < 1$, for $s = 1(1)l$,

(ii) if t_s is complex, then $t_{s+1} = \bar{t}_s$.

To each t_s in T_l, we associate a number ϵ_s where ϵ_s is either $+1$ or -1. We let Σ_l denote the set $\{\epsilon_1, \epsilon_2, ..., \epsilon_l\}$ with the restriction that if t_s is complex then $\epsilon_{s+1} = \epsilon_s$. When $l = 0$, T_l and Σ_l are taken to be null sets.

To each t_s in T_l, we also associate a set of integers $\{a_{j,s}\}$ which may be finite or infinite. If t_s is complex, then we shall impose the condition that the set $\{a_{j,s+1}\}$ is the same as the set $\{a_{j,s}\}$. We shall denote by A the set of integers $\{a_{j,s}\}$ taken over all j and all s.

DEFINITION 2.3. For $\theta \in [0, \pi]$, we define the function $\Delta_j(A, T_l, \Sigma_l; \theta)$ by

$$\Delta_j(A, T_l, \Sigma_l; \theta) = \sum_{s=1}^{l} \epsilon_s \delta(a_{j,s}, t_s; \theta). \tag{2.2}$$

In places where no confusion occurs, we shall write $\Delta_{l,j}$ for $\Delta_j(A, T_l, \Sigma_l; \theta)$.

LEMMA 2.2. *$\Delta_{l,j}$ has the following properties*:

(i) *$\Delta_{l,j}$ is a real continuous function of θ on* $[0, \pi]$.

(ii) $\Delta_j(A, T_l, \Sigma_l; 0) = 0$ *and* $\Delta_j(A, T_l, \Sigma_l; \pi) = \sum_{s=1}^{l} \epsilon_s a_{j,s}\pi$.

Proof follows directly from Lemma 2.1.

DEFINITION 2.4. For any integer k, we define a real function $F_k(A, T_l, \Sigma_l; x)$ on $[-1, 1]$ by

$$F_k(A, T_l, \Sigma_l; x) = \cos[k\theta + \Delta_j(A, T_l, \Sigma_l; \theta)], \tag{2.3}$$

where $x = \cos\theta$.

These functions can be looked upon as generalizations of the Chebyshev polynomials of the first kind. For when $l = 0$, $F_k(A, T_0, \Sigma_0; x) = T_{|k|}(x)$. It is worth noting that although we have introduced $F_k(A, T_l, E_l; x)$ as in (2.3), this class of functions has also been used by other authors under different notations (see, for example, Meinardus [4], p. 38). We shall now obtain some properties of $F_k(A, T_l, \Sigma_l; x)$.

LEMMA 2.3. *$F_k(A, T_l, \Sigma_l; x)$ attains its extreme values of ± 1 alternately in at least* $(1 + |k + \sum_{s=1}^{l} \epsilon_s a_{j,s}|)$ *points of* $[-1, 1]$.

Proof. As θ varies from 0 to π, $\{k\theta + \Delta_{l,j}\}$ takes all values in $[0, (k + \sum_{s=1}^{l} \epsilon_s a_{j,s})\pi]$, since $\Delta_{l,j}$ is a continuous function of θ in $[0, \pi]$. The result follows immediately.

LEMMA 2.4. *$F_k(A, T_l, \Sigma_l; x)$ is a quotient of two polynomials of degree* $(|k| + \sum_{s=1}^{l} |a_{j,s}|)$ *and* $(\sum_{s=1}^{l} |a_{j,s}|)$, *respectively*.

Proof. We can assume without loss of generality that $\epsilon_s = +1$ for $s = 1(1)\, l_1$ and -1 for $s = (l_1 + 1)(1)(l)$. Then $F_k(A, T_l, \Sigma_l; x)$ can be written in the form

$$F_k(A, T_l, \Sigma_l; x) = \operatorname{Re}\left\{e^{ik\theta} \prod_{s=1}^{l_1} \left[\frac{t_s e^{ia_{j,s}\theta} - 1}{t_s - e^{ia_{j,s}\theta}}\right] \prod_{s=(l_1+1)}^{l} \left[\frac{t_s - e^{ia_{j,s}\theta}}{t_s e^{ia_{j,s}\theta} - 1}\right]\right\}.$$

On multiplying both the numerator and denominator by the complex conjugate of the latter, we observe that the right-hand side is a quotient of two polynomials of degree $(|k| + \sum_{s=1}^{l} |a_{j,s}|)$ and $(\sum_{s=1}^{l} |a_{j,s}|)$, in $\cos\theta$, respectively.

We are now ready to introduce a special class of continuous functions. Let $\{k_j\}_{j=0}^{\infty}$ be a subsequence of nonnegative integers and $\lambda = \{\lambda_j\}_{j=0}^{\infty}$ be any sequence of arbitrary numbers such that $\sum_{j=0}^{\infty} |\lambda_j|$ is finite. We define a function $f(A, T_l, \Sigma_l, \lambda; x)$ by

$$f(A, T_l, \Sigma_l, \lambda; x) = \sum_{j=0}^{\infty} \lambda_j F_{k_j}(A, T_l, \Sigma_l; x), \tag{2.4}$$

and obtain the following result.

LEMMA 2.5. *$f(A, T_l, \Sigma_l, \lambda; x)$ is continuous on* $[-1, 1]$.

Proof. Since $|F_{k_j}(A, T_l, \Sigma_l; x)| \leqslant 1$ for every $x \in [-1, 1]$ and $\sum_{j=0}^{\infty} |\lambda_j|$ is finite, the series $\sum_{j=0}^{\infty} \lambda_j F_{k_j}(A, T_l, \Sigma_l; x)$ converges uniformly on $[-1, 1]$. Hence, f is a continuous function.

We shall now obtain an extension of a result by Rivlin [5], using appropriate choices of the sets A, Σ_l and the sequences $\{\lambda_j\}$, $\{k_j\}$.

3. A GENERALIZATION OF A RESULT DUE TO RIVLIN

Let $a_{j,s} = a_s$, for $s = 1(1)l$ and all j, so that A is a finite set of l integers. Also, let $\epsilon_s = +1$, for $s = 1(1)l$, and $\lambda_j = \gamma^j$, where γ is a real number satisfying $|\gamma| < 1$. If we choose $k_j = aj + b$, where a, b are positive integers, $a \geqslant 1$ and $b \geqslant 0$, then the function f can be expressed as follows.

LEMMA 3.1.

$$f = (F_b - \gamma F_{b-a})/[1 + \gamma^2 - 2\gamma T_a(x)], \tag{3.1}$$

where F_b and F_{b-a} denote $F_b(A, T_l, \Sigma_l; x)$ and $F_{b-a}(A, T_l, \Sigma_l; x)$, respectively.

Proof. From (2.4), we have

$$f = \sum_{j=0}^{\infty} \gamma^j F_{aj+b}(A, T_l, \Sigma_l; x),$$

which can be written as

$$f = \text{Re}\left\{e^{i[b\theta + \Delta_{l,j}]} \sum_{j=0}^{\infty} (\gamma e^{ia\theta})^j\right\}.$$

Identity (3.1) is then obtained by summing the infinite series.

For any nonnegative integer p, we define the function $f_p(A, T_l, \Sigma_l, \lambda; x)$ by

$$f_p(A, T_l, \Sigma_l, \lambda; x) = \sum_{j=0}^{p} \gamma^j F_{aj+b} + \frac{\gamma^{p+2}}{1-\gamma^2} F_{ap+b}. \tag{3.2}$$

THEOREM 3.1. *Let a_s be positive integers such that $\sum_{s=1}^{l} a_s \leqslant a - 1$, and let n be any integer satisfying*

$$\sum_{s=1}^{l} a_s \leqslant n \leqslant a - 1. \tag{3.3}$$

Let m be any nonnegative integer, and suppose p is such that

$$ap + b + \sum_{s=1}^{l} a_s \leqslant m \leqslant a(p+1) + b - 1. \tag{3.4}$$

Then $f_p(A, T_l, \Sigma_l, \lambda; x)$ is the best uniform rational approximation $r^*(m, n; x)$ to f and

$$E_{m,n}(f) = |\gamma|^{p+1}/(1-\gamma^2). \tag{3.5}$$

Proof. Let us write

$$\epsilon(x) = f(A, T_l, \Sigma_l, \lambda; x) - f_p(A, T_l, \Sigma_l, \lambda; x).$$

Then

$$\epsilon(x) = \operatorname{Re}\left\{e^{i(b\theta+\Delta_{l,j})}\left[\sum_{j=p+1}^{\infty} (\gamma e^{ia\theta})^j - \frac{\gamma^{p+2}}{1-\gamma^2} e^{iap\theta}\right]\right\},$$

$$= \operatorname{Re}\left\{\frac{\gamma^{p+1}}{1-\gamma^2} e^{i(b\theta+\Delta_{l,j})} e^{iap\theta}\left[\frac{e^{ia\theta}-\gamma}{1-\gamma e^{ia\theta}}\right]\right\},$$

on summing the series. Hence

$$\epsilon(x) = \frac{\gamma^{p+1}}{1-\gamma^2} F_{ap+b}(A, T_{l+1}, \Sigma_{l+1}; x), \tag{3.6}$$

where $\epsilon_{l+1} = +1$, $t_{l+1} = \gamma$, and $a_{l+1} = a$.

From Lemma 2.3, the function $F_{ap+b}(A, T_{l+1}, \Sigma_{l+1}; x)$ attains its extreme values ± 1 alternately in at least $(ap + b + \sum_{s=1}^{l} a_s + a + 1)$ points. Also, $f_p(A, T_l, \Sigma_l, \lambda; x)$ is a rational function of the form $\bar{r}(ap + b + \sum_{s=1}^{l} a_s, \sum_{s=1}^{l} a_s; x)$. Thus, by Lemma 1.1, f_p is the best uniform rational approximation $r^*(ap + b + \sum_{s=1}^{l} a_s + u, \sum_{s=1}^{l} a_s + v; x)$ to f, where $0 \leqslant u, v \leqslant a - \sum_{s=1}^{l} a_s - 1$. By putting

$$m = ap + b + \sum_{s=1}^{l} a_s + u \quad \text{and} \quad n = \sum_{s=1}^{l} a_s + v,$$

the conditions (3.3) and (3.4) follow. We have

$$E_{m,n}(f) = \max_{x\in[-1,1]} |\epsilon(x)| = \frac{|\gamma|^{p+1}}{1-\gamma^2}.$$

COROLLARY. *The results in Theorem* 3.1 *can be readily extended to the case of the function* $(\alpha + \beta f)$, *where* α *and* β *are two arbitrary (real or complex) constants.*

Comment. If $l = 0$; a, b are two integers such that $a \geqslant 1$, $b \geqslant 0$ and $|\gamma| < 1$, then

$$f = \frac{T_b(x) - \gamma T_{|b-a|}(x)}{1 + \gamma^2 - 2\gamma T_a(x)}. \tag{3.7}$$

Let n be an integer satisfying $0 \leqslant n \leqslant a - 1$ and m be any given nonnegative integer, then $r^*(m, n; x)$ is a polynomial of degree $(ap + b)$, where p is such that $ap + b \leqslant m \leqslant a(p + 1) + b - 1$.

$$r^*(m, n; x) = \sum_{j=0}^{p} \gamma^j T_{aj+b} + \frac{\gamma^{p+2}}{1-\gamma^2} T_{ap+b}, \tag{3.8}$$

$$E_{m,n}(f) = |\gamma|^{p+1}/(1 - \gamma^2). \tag{3.9}$$

In particular, when $n = 0$, $r^*(m, 0; x) = p_m^*(x)$, $E_{m,0}(f) = E_m(f)$, and we obtain precisely Rivlin's result [5] for the case of polynomial approximation.

4. A GENERALIZATION OF A RESULT DUE TO BERNSTEIN

Let l be an even integer, $\epsilon_s = +1$ for $s = 1(1)(l/2)$ and $\epsilon_s = -1$ for $s = (l/2 + 1)(1)(l)$. Let $\{k_j\}_{j=0}^{\infty}$ be such that the ratios k_{j+1}/k_j, $j = 0, 1, 2,...$, are odd integers $>(2l + 1)$. Furthermore, we choose $a_{j,s} = k_j$, for $s = 1(1)l$, $j = 0, 1, 2,...$, and let $\{\lambda_j\}_{j=0}^{\infty}$ be a sequence of nonnegative real numbers such that $\sum_{j=0}^{\infty} \lambda_j$ is finite. For any nonnegative integer p, we define the function $f_p(A, T_l, \Sigma_l, \lambda; x)$ by

$$f_p(A, T_l, \Sigma_l, \lambda; x) = \sum_{j=0}^{p} \lambda_j F_{k_j}(A, T_l, \Sigma_l; x). \tag{4.1}$$

THEOREM 4.1. *Given two nonnegative integers* m, n, *suppose* p *is such that*

$$(l + 1)\, k_p \leqslant m \leqslant k_{p+1} - lk_p - 1,$$

and (4.2)

$$lk_p \leqslant n \leqslant k_{p+1} - (l + 1)\, k_p - 1.$$

Then $f_p(A, T_l, \Sigma_l, \lambda; x)$ is the best uniform rational approximation $r^*(m, n; x)$ to f and

$$E_{m,n}(f) = \sum_{j=p+1}^{\infty} \lambda_j .$$

Proof. Let us write

$$\epsilon(x) = f(A, T_l, \Sigma_l, \lambda; x) - f_p(A, T_l, \Sigma_l, \lambda; x),$$

$$= \sum_{j=p+1}^{\infty} \lambda_j F_{k_j}(A, T_l, \Sigma_l; x). \tag{4.3}$$

We want to consider the values of $\epsilon(x)$ at $(k_{p+1} + 1)$ points x_q of $[-1, 1]$, where

$$x_q = \cos\theta_q = \cos(q\pi/k_{p+1}), \qquad q = 0(1)(k_{p+1}).$$

Then,

$$\epsilon(x_q) = \sum_{j=p+1}^{\infty} \lambda_j \cos\left(k_j\theta_q + \sum_{s=1}^{l} \epsilon_s \delta(k_j, t_s; \theta_q)\right).$$

But from Definition 2.1, we have

$$\delta(k_j, t_s; \theta_q) = \delta\left(\frac{qk_j}{k_{p+1}}, t_s; \pi\right) = \frac{qk_j}{k_{p+1}}\pi.$$

This is independent of s, so that

$$\sum_{s=1}^{l} \epsilon_s \delta(k_j, t_s; \theta_q) = \frac{qk_j\pi}{k_{p+1}} \sum_{s=1}^{l} \epsilon_s = 0.$$

Hence,

$$\epsilon(x_q) = \sum_{j=k+1}^{\infty} \lambda_j \cos\left(\frac{qk_j}{k_{p+1}}\pi\right) = (-1)^q \sum_{j=p+1}^{\infty} \lambda_j ,$$

as k_j/k_{p+1} is an odd integer whenever $j > (p + 1)$. Thus, $\epsilon(x)$ attains its extrema alternately in at least $(k_{p+1} + 1)$ points in $[-1, 1]$. From Lemma 2.4, we find that $f_p(A, T_l, \Sigma_l, \lambda; x)$ is a rational function of the form $\bar{r}((l + 1)k_p, lk_p; x)$. By using the result of Lemma 1.1, f_p is the best uniform rational approximation $r^*((l + 1)k_p + u, lk_p + v; x)$ to f, where $0 \leqslant u,\ v \leqslant k_{p+1} - (2l + 1)k_p - 1$. We now put $m = (l + 1)k_p + u$, $n = lk_p + v$, and obtain the conditions (4.2) for p.

Comments.

(i) When $l = 0$, the Chebyshev series

$$f(A, T_0, \Sigma_0, \lambda; x) = \sum_{j=0}^{\infty} \lambda_j T_{k_j}(x) \tag{4.4}$$

has its truncated series $f_p(A, T_0, \Sigma_0, \lambda; x)$ of degree p as its best uniform rational approximation $r^*(m, n; x)$ if p satisfies $k_p \leqslant m \leqslant k_{p+1} - 1$ and $0 \leqslant n \leqslant k_{p+1} - 1$. In particular, when $n = 0$, we obtain a result for the best uniform polynomial approximation given by Bernstein (see Golomb [3], p. 163).

(ii) As a special case of Theorem 4.1, choose $k_j = ab^j$, where a is a positive integer and b is an odd integer $>(2l + 1)$. We obtain a result which can be considered as an extension of a result due to Boehm [2] who considered essentially the case $l = 0$.

(iii) We now choose $k_j = a^j$, where a is an odd integer $>(2l + 1)$, and $\lambda_j = \gamma^j$, where γ is a real number satisfying $0 < \gamma < 1$. When $l = 0$, $f(A, T_0, \Sigma_0, \lambda; x)$ is then the well-known Weierstrass function (see Achieser [1], p. 66). Putting $n = 0$, we obtain its best uniform polynomial approximation.

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